Induction Introduction Problems

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1 Problems

1.1 Problem 1

Use induction to show that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ for all positive integers n.

1.2 Problem 2

Use induction to prove that $5^n - 1$ is divisible by 4 for all non-negative integers n.

1.3 Problem 3

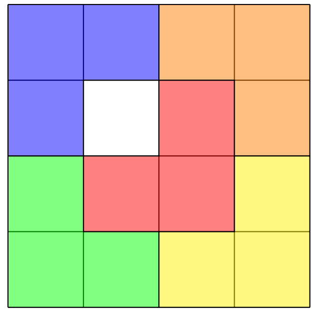
Define a sequence with $a_1 = 1$ and $a_n = \sqrt{3a_{n-1} + 8}$. Prove that $a_n < 5$ for all positive integers n.

1.4 Problem 4

(Duplicate, this problem is on the slides as well.)

Prove that for all positive integers n, x, y with $x, y \leq 2^n$, it is possible to tile a $2^n \times 2^n$ grid with L-shaped blocks consisting of 3 unit squares, such that only the cell at position (x, y) is not covered.

For example, when $n = 2, x = 2, y = 3$, the following tiling can be used.



1.5 Problem 5

Prove that in any tournament graph, there exists a Hamiltonian path.

Definitions

A tournament graph is a directed graph containing N nodes and $\frac{N(N-1)}{2}$ edges, for some positive integer N, such that between every pair of distinct nodes, there is exactly one directed edge, which might be directed in either of the two possible orientations.

A directed graph $G = (V, E)$, consists of a set of nodes (V) and a set of edges (E). For simplicity, assume the nodes are numbered with the positive integers $1, 2, \ldots, N$. Each edge is an ordered pair of nodes. For an edge (u, v), it is said that it is directed from u to v.

A Hamiltonian path in a graph is a path that visits each node exactly once.

A path P of length k in a graph is a list of nodes $P_1, P_2, \ldots, P_k$ such that there is an edge from each node in the path to the next. I.e. the graph contains the edge (P_i, P_{i+1}) for all $1 \leq i \leq k - 1$.

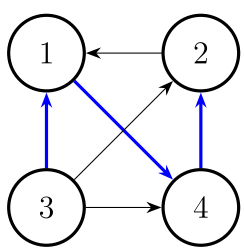


Figure 1: An example of a tournament graph of size 4 with a Hamiltonian path highlighted.

1.6 Problem 6

Prove that for each positive integer, there is at most one way of writing it as the sum of distinct, non-consecutive Fibonacci numbers.

The Fibonacci sequence is defined as $F(1) = 1, F(2) = 2, F(n) = F(n-1) + F(n-2)$.

Formally, prove that for any positive integer N, there exists at most one set $S = \{S_1, S_2, \ldots, S_k\}$ such that $S_i \in \mathbb{Z}^+$, $\sum_{i=1}^{k} F(S_i) = N$ and $|S_i - S_j| > 1$ for all $1 \leq i < j \leq k$.

2 Solutions

2.1 Solution for Problem 1

For $n = 1$, $1 = \frac{1(2)}{2}$.

For $n \geq 2$, assume $1 + 2 + \ldots + n - 1 = \frac{(n-1) \cdot n}{2}$.

Then,

$$\begin{aligned}
1 + 2 + \ldots + n - 1 &= \frac{(n-1) \cdot n}{2} \\
1 + 2 + \ldots + n - 1 + n &= \frac{n(n-1)}{2} + n \\
1 + 2 + \ldots + n &= \frac{n(n-1) + 2n}{2} \\
1 + 2 + \ldots + n &= \frac{n(n+1)}{2}
\end{aligned}$$

This completes the induction.

2.2 Solution for Problem 2

$5^0 - 1 = 0$ is divisible by 4.

For $n \geq 1$, assume that $5^{n-1} - 1$ is divisible by 4. Another way of saying this is that there exists an integer k such that $5^{n-1} - 1 = 4k$.

Next, we rearrange:

$$\begin{aligned} 5^{n-1} - 1 &= 4k \\ 5^n - 5 &= 20k \\ 5^n - 1 &= 20k + 4 \\ 5^n - 1 &= 4(5k + 1) \end{aligned}$$

Since $5k + 1$ is an integer, $5^{n-1} - 1$ being divisible by 4 implies that $5^n - 1$ is also divisible by 4, completing the induction.

2.3 Solution for Problem 3

We can use induction to show that $a_n < 5$. $a_1 = 1 < 5$. For $n \geq 2$, assume that $a_{n-1} < 5$.

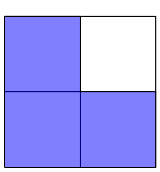
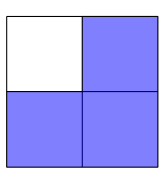
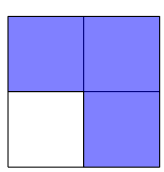
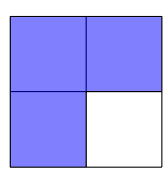
$$\begin{aligned} a_{n-1} &< 5 \\ 3a_{n-1} &< 15 \\ 3a_{n-1} + 8 &< 23 \\ \sqrt{3a_{n-1} + 8} &< \sqrt{23} \\ a_n &< \sqrt{23} < 5 \end{aligned}$$

This completes the induction, showing that $a_n < 5$.

2.4 Solution for Problem 4

Assume that it is possible to construct a tiling that covers all but any given cell for a $2^n \times 2^n$ grid.

This is obviously true for $n = 1$ since you can use the 4 rotations of the block to leave whichever square you want uncovered.

We will prove that if it is possible for $n-1$, it is also possible for n. Find which quadrant the cell that should be empty is on, and tile that quadrant so that only that cell is uncovered. This is possible since it is the $n-1$ case. For all the other quadrants, tile them such that only the corners of the quadrants that are in the middle of the larger square are not covered, This is also possible since the quadrants have size $2^{n-1} \times 2^{n-1}$. Cover the remaining 3 cells in the center of the larger square with a single L-shaped block. Now all cells but the required one are covered, completing the induction.

2.5 Solution for Problem 5

We will use proof by induction to show that there is a Hamiltonian path on a tournament graph of size N.

The $N = 1$ case is trivially true, the path consists of only the one node that exists.

For all $N \geq 2$, assume there exists a Hamiltonian path in every tournament graph of size $N-1$.

Pick any node u in the graph, and construct a Hamiltonian path P on the remaining $N-1$ nodes.

If the graph contains the edge (u, P_1), u can be added to the start of P to construct the path for the full graph.

If the graph contains the edge (P_{N-1}, u), u can be added to the end of P to construct the new path.

The only remaining case is when the graph has edges (P_1, u) and (u, P_{N-1}). In this case, find the smallest index i such that there is an edge (u, P_i). It is guaranteed that such an index exists since the graph has the edge (u, P_{N-1}). It is guaranteed that $i > 0$ since the graph does not have the edge (u, P_1). It is guaranteed that the graph has the edge (P_{i-1}, u), since the graph contains every edge in one of the two possible orientations, and if it was in the other orientation, we would've chosen index $i-1$, not index i. Thus, we can add node u to the path between indices $i-1$ and i, forming a new valid path, since the graph has edges (P_{i-1}, u) and (u, P_i).

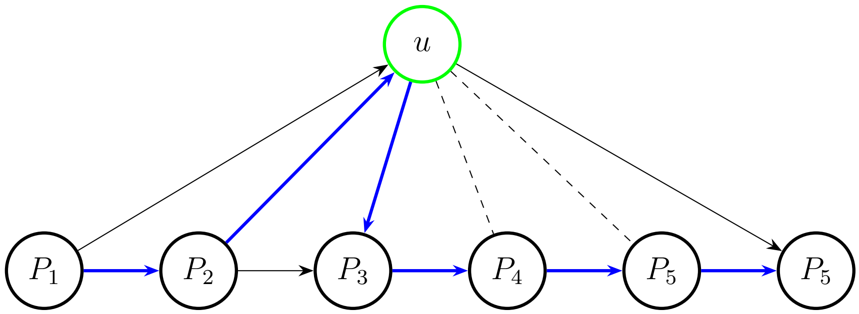


Figure 2: An example of the last case of the inductive step with $i = 3$.

This completes the induction, proving that if there is a Hamiltonian path in every tournament graph of size $N-1$, there is also a Hamiltonian path in every tournament graph of size N.

Since we already proved the $N = 1$ case, we conclude that there is a Hamiltonian path in every tournament graph.

2.6 Solution for Problem 6

Let $s(n)$ be the maximum sum you can obtain using the first n Fibonacci numbers if you can only use a subset of them that doesn't contain consecutive Fibonacci numbers. $s(0) = 0$. $s(1) = 1$. $s(2) = 2$. $s(n) = F(n) + s(n-2)$ for $n \geq 2$.

We will first show that $s(n-1) = F(n) - 1$ for $n \geq 1$. This is trivially true for $n = 1$ and $n = 2$. For $n \geq 3$, assume the statement is true for $n-2$.

$$s(n-1) = F(n-1) + s(n-3)$$
$$s(n-1) = F(n-1) + F(n-2) - 1$$
$$s(n-1) = F(n) - 1$$

This completes the induction.

Next, we use induction to show that for any non-negative integer x, there is at most one way of writing it as the sum of distinct, non-consecutive Fibonacci numbers. There is obviously only one subset of the Fibonacci numbers that sums to 0, the empty set, so this is true for $x = 0$.

For larger x, assume we have already proven that there is at most one valid sum for all integers less than x. Note that the largest Fibonacci number $F(n)$ less than or equal to x must be included in the sum for x. This is because if we didn't include $F(n)$, the largest sum we could make is $s(n-1) = F(n)-1$ which is strictly less than x. This means that to construct the sum for x, we must include $F(n)$ and the sum for $x - F(n)$, which we already know is unique since $x - F(n) < x$. This completes the induction, proving that there is at most one valid sum for x.

Note that this does not guarantee that there always exists at least one way of writing a sum for each x, since it's possible that the sum for $x - F(n)$ requires $F(n - 1)$, making the sum invalid.