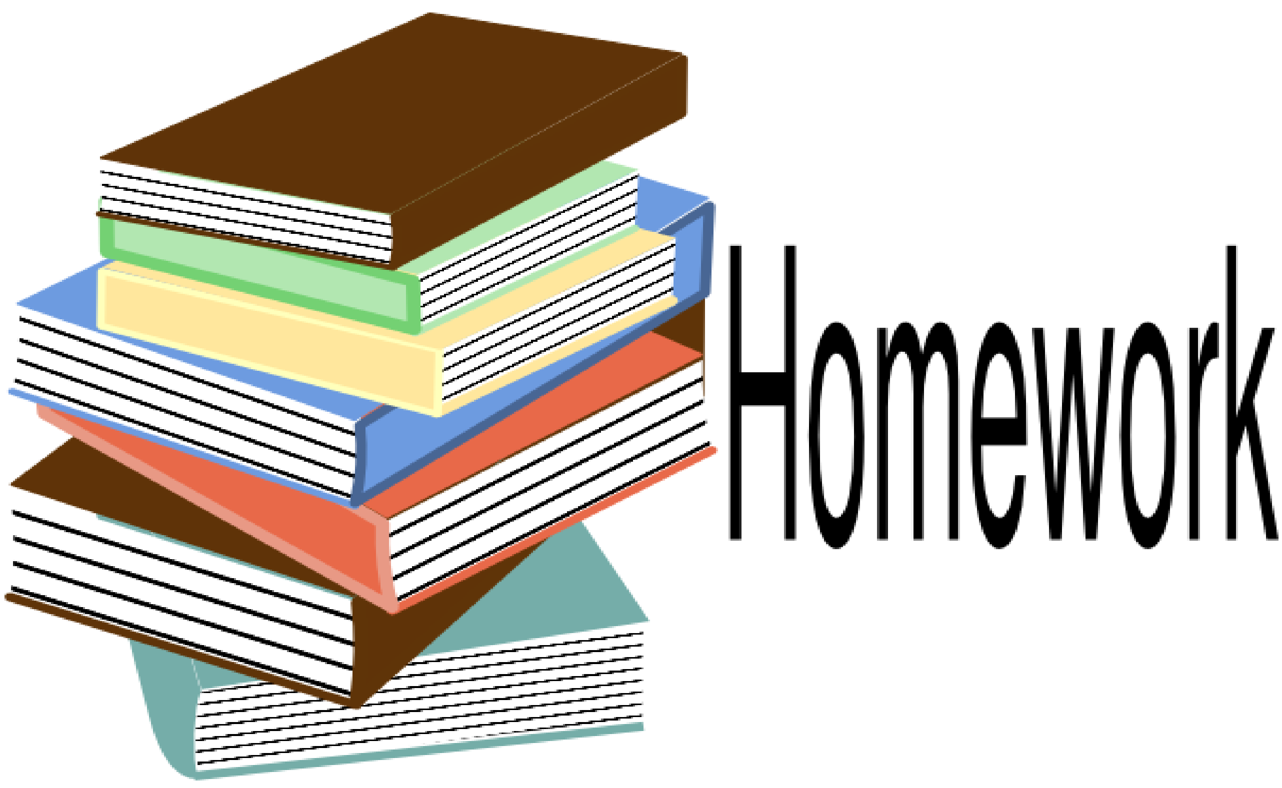
INDIRECT METHODS OF PROOF

Problems for the class



Problem 1

<u>Statement:</u>
For any integers a and b, $a + b \geq 15$ implies that $a \geq 8$ or $b \geq 8$.

<u>Prove by contrapositive:</u>

TRY BY YOURSELF!

Then
Check the solution on the next page.

Problem 1. Solution.

<u>Statement:</u>
For any integers a and b, $a + b \geq 15$ implies that $a \geq 8$ or $b \geq 8$.

<u>Proof by contrapositive:</u>

We'll prove the contrapositive of this statement, that is, for any integers a and b, $a < 8$ and $b < 8$ implies that $a + b < 15$.

So, suppose that a and b are integers such that $a < 8$ and $b < 8$.

Since they are integers (not e.g. real numbers), this implies that $a \leq 7$ and $b \leq 7$.

Adding these two equations together, we find that
$a + b \leq 14$.

But this implies that $a + b < 15$, which completes the proof by contrapositive.

Problem 2

Statement:

Let x be a real number. If $x^3 - 7x^2 + x - 7 = 0$ then $x=7$

Prove by contrapositive:

TRY BY YOURSELF!

Then
Check the solution on the next page.

Problem 2. Solution

<u>Prove by contrapositive:</u>

Assume $x \neq 7$

Then
$x^3 - 7x^2 + x - 7 \neq 0$

We have
$x^2 (x - 7) + (x -7) \neq 0$

$(x^2 +1)(x - 7) \neq 0$

So, $x^2 +1 = 0$ or $x - 7 = 0$

We got $x = 7$

We proved that if $x^3 - 7x^2 + x - 7 = 0$ then $x=7$

Problem 3

Statement:

$a^2 - 4b \neq 2$ when $a, b \in Z$

Prove by contradiction:

TRY BY YOURSELF!

Then
Check the solution on the next page.

Problem 3. Solution.

Prove by contradiction:

Assume $a^2 - 4b = 2$ for integers a and b.

We have
$a^2 = 4b + 2$
$a^2 = 2(2b + 1)$

Let $2b + 1 = c$,
So, $a^2 = 2c$
That means that a^2 is even.
As a^2 is even, so a is also even.

By definition of even integers, $a=2k$, for some integer k.
After substitution $a=2k$ into the equation $a^2 - 4b = 2$
We get

$(2k)^2 - 4b = 2$
$4k^2 - 4b = 2$
$4(k^2 - b) = 2$

$k^2 = b + \frac{1}{2}$
But this is a contradiction, since k is integer.
Therefore, $a^2 - 4b \neq 2$ when $a, b \in Z$

Problem 4

Statement:

If n^2 is an odd integer, then n must be odd.

Prove by contradiction:

TRY BY YOURSELF!

Then
Check the solution on the next page.

Problem 4. Solution.

<u>Prove by contradiction:</u>

To prove by contradiction we would assume that the statement is not true.
Let's assume that there is an even integer n for which n^2 is odd.

Let $n = 2k$

$n^2 = (2k)^2$
$\quad = 4k^2$
$\quad = 2(2k^2)$

$2(2k^2)$ is always even as it is a multiple of two.

We now have a contradiction in our assumption that an even integer n exists where n^2 is odd.

Therefore if n^2 is an odd integer, then n must be odd.

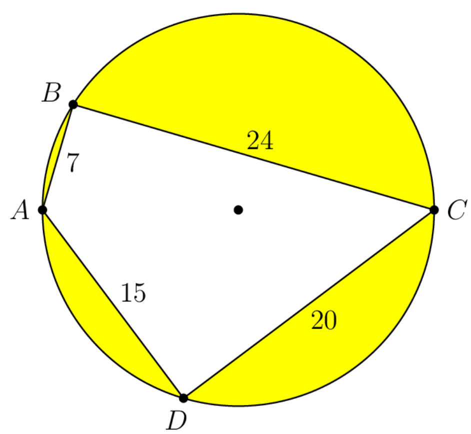
Problem 5

Quadrilateral ABCD with side lengths AB = 7, BC = 24, CD = 20, DA = 15 is inscribed in a circle.

The area interior to the circle but exterior to the quadrilateral can be written in the form $\frac{a\pi - b}{c}$, where a,b and c are positive integers such that a and c have no common prime factor.

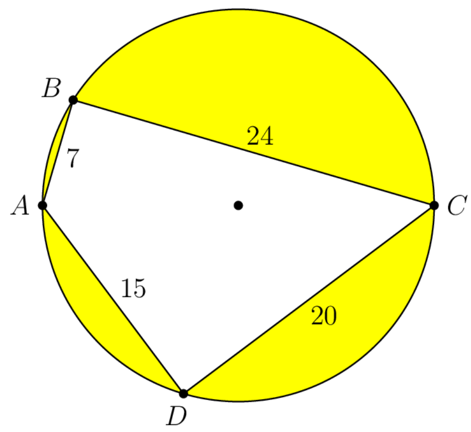
What is a + b + c?

$\textbf{(A)}\ 260 \qquad \textbf{(B)}\ 855 \qquad \textbf{(C)}\ 1235 \qquad \textbf{(D)}\ 1565 \qquad \textbf{(E)}\ 1997$



You can use Indirect methods of proof to solve this problem.

Problem 5. Solution.



Based on Inscribed Angle Theorem

Opposite angles of every cyclic quadrilateral are supplementary, so $\angle B + \angle D = 180°$.

$$AC = 25.$$

We claim that We can prove it by contradiction:

- If $AC < 25,$ then $\angle B$ and $\angle D$ are both acute angles. This arrives at a contradiction.
- If $AC > 25,$ then $\angle B$ and $\angle D$ are both obtuse angles. This arrives at a contradiction.

By the Inscribed Angle Theorem, we conclude that $\overline{AC}$ is the diameter of the circle. So, the radius of the circle is

$r = \dfrac{AC}{2} = \dfrac{25}{2}.$

The area of the requested region is

$$\pi r^2 - \frac{1}{2} \cdot AB \cdot BC - \frac{1}{2} \cdot AD \cdot DC = \frac{625\pi}{4} - \frac{168}{2} - \frac{300}{2} = \frac{625\pi - 936}{4}.$$

Therefore, the answer is

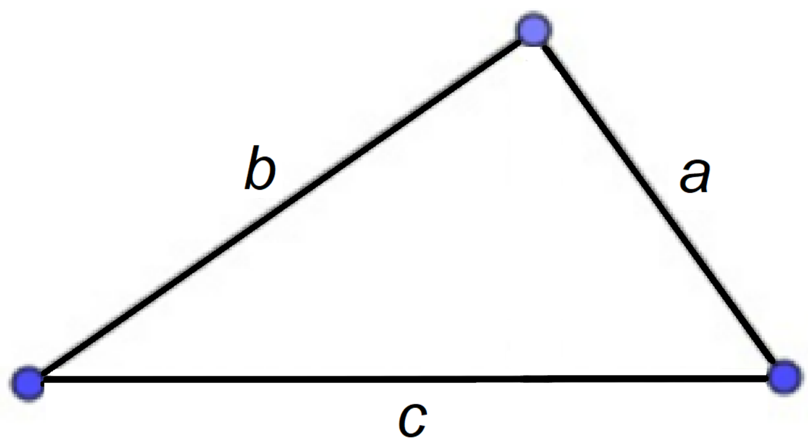
$$a + b + c = \boxed{\textbf{(D) } 1565}.$$

Problem 6

Statement:

Prove the Pythagorean theorem.

Given a right triangle with the side lengths a, b, and the hypotenuse c,



Then, $a^2 + b^2 = c^2$.

Prove by contradiction:

TRY BY YOURSELF!

Then
Check the solution on the next page.

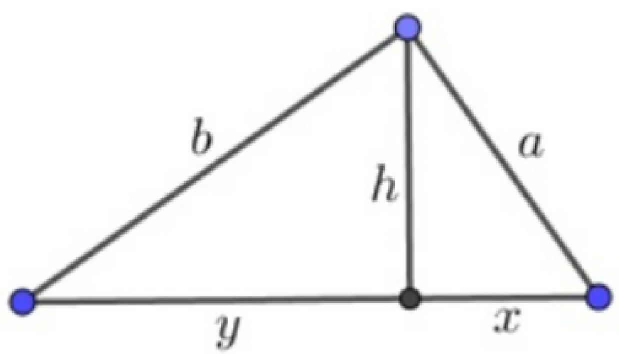
Problem 6. Solution.

<u>Statement:</u>

Given a right triangle with the side lengths a, b, and the hypotenuse c, $a^2 + b^2 = c^2$.

<u>Prove by contradiction:</u>

Let h be the altitude from the right angle. The right triangle (a, b, c) is split into two triangles, (x, h, a) and (h, y, b), both similar to the triangle (a, b, c). In particular, $h^2 = xy$.



c = x+y

Assume to the contrary that $a^2 + b^2 \neq c^2$. Let $a^2 + b^2 \prec c^2$. Then, by similarity also $h^2 + y^2 \prec b^2$ and $h^2 + x^2 \prec a^2$

Summing up yields a contradiction:

$a^2 + b^2 \succ h^2 + y^2 + h^2 + x^2 = 2h^2 + y^2 + x^2 = 2xy + y^2 + x^2 =$
$= (x + y)^2 = c^2$,

so that $a^2 + b^2 \succ c^2$ is contrary to our assumption.

Therefore, $a^2 + b^2 = c^2$.

References:

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